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Scalar-QED $\hbar$-corrections to the coulomb potential

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ABSTRACT: The leading long-distance 1-loop quantum corrections to the Coulomb potential are derived for scalar QED and their gauge-independence is explicitly checked. The potential is obtained from the direct calculation of the 2-particle scattering amplitude, taking into account all relevant 1-loop diagrams. Our investigation should be regarded as a first step towards the same programme for effective Quantum Gravity. In particular, with our calculation in the framework of scalar QED, we are able to demonstrate the incompleteness of some previous studies concerning the Quantum Gravity counterpart.

KEYWORDS: Renormalization Regularization and Renormalons, Models of Quantum Gravity, Supersymmetric Effective Theories, Electromagnetic Processes and Properties.

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1. Introduction

The non-renormalizability of General Relativity [1, 2] has inspired the study of various alternative models for Quantum Gravity. It was soon realized that proposals based on high-derivative local field theories are inconsistent with the simultaneous requirements of unitarity and renormalizability [3] (see also [4] for the discussion of the spectrum of a general high-derivative local Quantum Gravity) and the main attention is now concentrated on non-local objects like strings and p-branes. Therefore, while high energy effects are described by strings, at low energies one meets effective gravity actions, which may be the Einstein-Hilbert one or extensions based on the inclusion of some additional fields, such as a dilaton. It was already noticed by S. Weinberg, in [5], that Quantum Gravity based on General Relativity may be consistent as a quantum theory for the restricted low-energy domain. The idea of an effective approach to Quantum Gravity was realized in a recent paper by F. Donoghue [6], who used it to perform a practical calculation of the leading long-distance quantum correction to the Newton potential.[1] (See also [8] for a general explanation of the effective approach). This work attracted considerable interest and raised the hopes to apply the background of effective Quantum Gravity to other problems. However, as concerning the original calculations in Quantum Gravity [6], some questions still remain unanswered.

[1]A similar calculation has been done long ago by Iwasaki [7].

The calculations imply two important suppositions. First, one has to deal with the separation of the long-distance effects, related to the non-local part of the effective action, from the UV divergent pieces, which may always be subtracted by adding local counterterms. Indeed, all these counterterms have plenty of higher derivatives, but they may be removed by the renormalization of the corresponding high derivative terms in the action. These terms, in turn, are invisible at low energies, because their corresponding degrees of freedom have huge masses. Second, some part of the long-distance contributions of the Feynman diagrams are proportional to the well-known UV divergences, and this enables one to greatly reduce the volume of calculations [6]. This part is composed by the logarithmic (L-type in [6]) non-analytic terms, and they come from the diagrams with only massless internal lines. At the same time, there are other one-loop diagrams which produce, along with L-type structures, other (S-type in [6]) non-analytic terms, and those are absolutely independent from the UV divergences. Indeed, this kind of terms gives the leading contribution to the long-distance quantum corrections. After the original papers [6] were published, there was a series of works devoted to their correction and checking. In particular, [9] pointed out the error in the calculation of the one-loop contribution to the vertex function, while [10] found some relevant diagrams which were not accounted for in [6].[2] At the same time, in the works of refs. [11, 12] functional methods were applied for the same calculation. However, they succeeded in extracting only the L-type non-analytic terms, which can be easily obtained from the 1-loop logarithmic divergences. The leading S-type non-localities do not come out in this way, and it is still necessary suplement the programme by calculating diagrams. One has to notice that the S-type non-localities appear due to the mixed loops with both massive and massless internal lines, and that the corresponding diagrams are not subject to the Appelquist-Carazzone theorem [13]. Indeed, the mixed diagrams produce L-type nonlocalities, too. Thus, at present, we have some set of alternative results for the same quantity (quantum corrections to the Newton potential), and they do not fit with each other.

Another subtle point in all the procedure is the gauge-independence of the result. For instance, the original calculation of [6] has used the polarization operator obtained in [1], but this is well-known to be gauge-dependent [14, 15]. On the other hand, one can argue that, being related to a scattering amplitude, quantum corrections to the potential should be gauge-independent. Thus, one expects that the gauge-fixing dependence of the polarization operator must cancel the one coming from the vertex. However, the non-standard aspects of the above effective scheme

[2]Unfortunately, the authors of [10] did not provide sufficient details of their calculations. Here, we analyse the same diagrams as the ones considered in [10]. Nevetheless, our considerations are more general in that we adopt a general gauge-fixing condition, and also present technical details. From our point of view, such a calculation is essentially non-trivial and its details should be manifest.

make an explicit check of the gauge-independence reasonable. We mention that such a check was successfully done in [11], but only for the L-type terms, while the question for S-type terms remains open.

Practical calculations in effective Quantum Gravity meet two kind of technical difficulties. First, there are problems with extracting the non-analytic pieces from the diagrams with both massless and massive internal lines (S-type terms). The complexity of this operation greatly increases with the number of massive insertions. The second problem is the huge amount of algebraic steps which are necessary for the calculations of diagrams in Quantum Gravity, especially in general non-minimal gauges. In view of this, in the present paper, we choose as our working model Scalar Quantum Electrodynamics (SQED), where the volume of algebraic work is essentially reduced, while the systematic of the approach is quite the same: SQED has almost all the diagrams that one meets in Quantum Gravity, and also these diagrams have the same power-counting for the IR divergences. Hence, in the course of our present calculations, one can learn better how to select relevant diagrams and contemporarily develop a technique for extracting the non-local pieces from these diagrams.

Our paper is organized as follows. In section 2, we present the backgrounds of the model, including the Feynman rules and a quick derivation of the classical Coulomb potential from the tree-level amplitude. Section 3 contains the full list of one-loop diagrams and their classification with respect to the low-energy contributions. In sections 4 and 5, we present the details of the diagrams, discuss the cancellation of their gauge-dependent parts and derive the quantum correction to the potential. Finally, we draw our Conclusions in section 6. An appendix follows, where we list the low-momentum behaviour of the relevant 1-loop integrals.

2. Feynman rules and the Coulomb potential

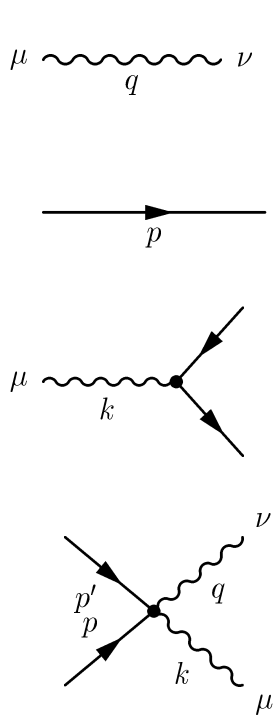
Let us consider SQED, starting from the non-invariant action with an arbitrary covariant gauge-fixing term:

$$S_{tot} = \int d^4x \left\{ -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{\alpha}\,(\partial_\rho A^\rho)^2 + \frac{1}{2} g^{\mu\nu}\,(D_\mu\phi)(D_\nu\phi)^* - \frac{1}{2} m^2\phi^*\phi \right\}. \qquad (2.1)$$

Here, α is the arbitrary gauge-fixing parameter. In the following, we shall consider the interaction between two heavy spinless particles of equal mass (m) and electric charge (e) as due to photon exchange. The effective interaction in the static regime is achieved by evaluating the scattering amplitude between these heavy particles in the limit of small momentum transfer, $q^2 \to 0$. This is readily carried out by using that, for a potential, V, the S-matrix element is given by

$$S = 1 - 2\pi\imath\,\delta(E_i - E_f)\langle f|V|i\rangle - 2\pi\imath\,\delta(E_i - E_f)\,\frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_i - E_n} + \cdots\,, \qquad (2.2)$$

where E_i, E_f are the energies of the initial and final asymptotic states, respectively.



$$= -\frac{i}{q^2}\left\{\eta_{\mu\nu} - (1+\alpha)\frac{q_\mu q_\nu}{q^2}\right\}$$

$$= \frac{i}{p^2 - m^2}$$

$$= -ie\left(p_\mu + p'_\mu\right)$$

$$= 2\, i\, e^2\, \eta_{\mu\nu}\,.$$

Figure 1: The Feynman rules for SQED.

If we denote the incoming and the outgoing momenta of the particles 1 and 2 by p_1, p_2 and p_3, p_4 respectively, and normalize the state vector as

$$\langle p_3\, p_4 | p_1\, p_2\rangle = (2\pi)^3 \delta^{(3)}(p_1 - p_3)(2\pi)^3\delta^{(3)}(p_2 - p_4)\,, \tag{2.3}$$

then

$$\langle p_3\, p_4 | V | p_1\, p_2\rangle = (2\pi)^3\delta^{(3)}(p_1 + p_2 - p_3 - p_4)\int V(r) e^{i\vec{q}\cdot\vec{r}} d^3 r\,, \qquad (\vec{q} = \vec{p}_1 - \vec{p}_3)\,. \tag{2.4}$$

In terms of the reactance matrix, the S-matrix may be parameterized as below:

$$S_{ij} = \delta_{ij} - i\,(2\pi)^4\,\delta^{(4)}(p_f - p_i) T(\vec{q})\,, \tag{2.5}$$

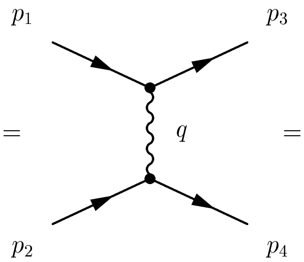
yielding the following expression for the potential:

$$V(r) = \frac{1}{(2\pi)^3}\int T(\vec{q})\, e^{-i\vec{q}\cdot\vec{r}}\, d^3\vec{r}\,. \tag{2.6}$$

As it was already mentioned, the advantage of working with SQED is that it leads to considerably simpler Feynman rules, while it presents diagrams similar to the ones appearing in the case of Quantum Gravity. The Feynman rules for the photon-matter vertices and propagators for an arbitrary gauge-fixing are given in figure 1.

The calculation of the tree-level graph at the static limit (figure 2) leads us to a scattering amplitude proportional to $1/q^2$, which gives rise to the classical Coulomb potential. By virtue of the current conservation, it can be readily checked that this result is completely independent of any gaugye-fixing procedure. The same can indeed be achieved by means of the above Feynman rules. The tree-level scattering amplitude has the form presented in figure 2.

$T(q) =$



$= \frac{i\,e^2}{q^2}\,.$

Figure 2: Tree level diagram, producing Coulomb potentialy

In the static limit, $q^0 = 0$, and therefore $q^2 = -\vec{q}^{\,2}$; so, after performing the Fourier transform

$$\int \frac{1}{\vec{q}^{\,2}}\, e^{i\vec{q}\cdot\vec{r}}\, d^3\vec{r} = \frac{1}{4\pi r}\,, \tag{2.7}$$

we obtain, as expected, the Coulomb potential:

$$V(r) = -\frac{e^2}{4\pi\, r}\,, \tag{2.8}$$

which is the tree-level approximation to the potential for the interaction between two static sources.

Before going on to specific calculations, it is possible to anticipate the form of the long distance $\hbar$-quantum corrections to the Coulomb potential, based upon dimensional analysis:

$$V(r) = -\frac{e^2}{r}\left(1 + \hbar\, e^2 \cdot \frac{\Gamma}{r} + \hbar\, e^2 \frac{\Lambda}{r^2}\right)\,, \tag{2.9}$$

where Γ and Λ are to be extracted from the loop diagrams and exhibit mass dimensions (-1) and (-2), respectively. They shall be shown to correspond to the L and S-types non-local 1-loop contributions.

Before closing this section and going on to calculate the relevant Feynman diagrams, we should perhaps make clear that the question of the potential in a field theory deserves some discussion. There is not a unique way of extracting a potential from a lagrangean field-theoretic model. In considering bound states or scattering processes, one should consider different ways of extracting a potental. This discussion becomes very clear in a series of papers by J. Sucher [16]

In our work, we shall be dealing with the potential as extracted from a scattering problem, as explained in the begining of this section. In the next section, we shall work out the quantum corrections to the scattering potential derived at tree-level, as illustrated above.

3. One-loop diagrams

Now, our task is to extract quantum corrections to the non-relativistic potential coming from quantum fluctuations of both the vector and scalar fields. Let us stress that the contributions from massive scalars cannot be disregarded, unless they form a closed loop without massless vector insertions. In fact, the only way to distinguish between the relevant and non-relevant diagrams is to check whether the particular graph has IR divergence, whenever the momentum transfer, $\vec{q}$, goes to zero. Only if the diagram has analytic behaviour in this limit, it can be left aside. This selection rule exactly corresponds to the result of [13]. In this point, we agree with some of the previous publications [6, 9, 10] and disagree with others: [11] and [12], where scalars were taken as merely classical sources. We shall consider the interaction induced by 1-loop contributions to the scattering amplitude between two heavy scalar particles of equal mass, m. The effective interaction in the static regime is then determined by evaluating the scattering amplitude, to order e^4, in the limit of very small momentum transfer, $\vec{q}^{\,2} \to 0$. In order to extract low-energy quantum corrections, the amplitudes are computed in momentum space as functions of the total momentum transfer, $\vec{q}^2$. To find such non-analytic terms, it is necessary to separate the UV finite pieces of the one-loop integrals, for instance by using dimensional regularisation [17]. This finite part contains all information concerning the low-energy behaviour of the amplitude, while the ultraviolet divergences have local structure and one disregards them, having in mind that they may be suitably renormalized.

In order to calculate the loop integrals, we have used the Feynman parametric representation for combining propagator denominators. The final answer then follows after performing the necessary momentum and parametric integrations, which are carried out by heavily making use of the software MAPLE V.

For small $\vec{q}^2$, the leading contributions arising from each diagram can then be separated into two types of terms, namely, $\ln(-\vec{q}^2)$ and $\pi^2\, m/\sqrt{\vec{q}^{\,2}}$, the latter coming exclusively from mixed massive-massless loops. The corrections to the potential, in coordinate space, come from momentum space calculations upon use of the Fourier transforms [6] similar to (2.7):

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{1}{\sqrt{\vec{q}^2}} &= \frac{1}{2\pi^2 r^2} \\ \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \ln \vec{q}^2 &= -\frac{1}{2\pi^2 r^3}\,. \end{aligned} \tag{3.1}$$

The full set of potentially relevant diagrams is presented in figure 3. One can classify all those diagrams using the non-analyticity as $q \to 0$ as a criterium. The first group of graphs includes $3a$, $3b$ and $3c$, and their respective permutations of external legs, which really contribute to the potential in the static limit. These diagrams will be considered in details in the next section. The second group is

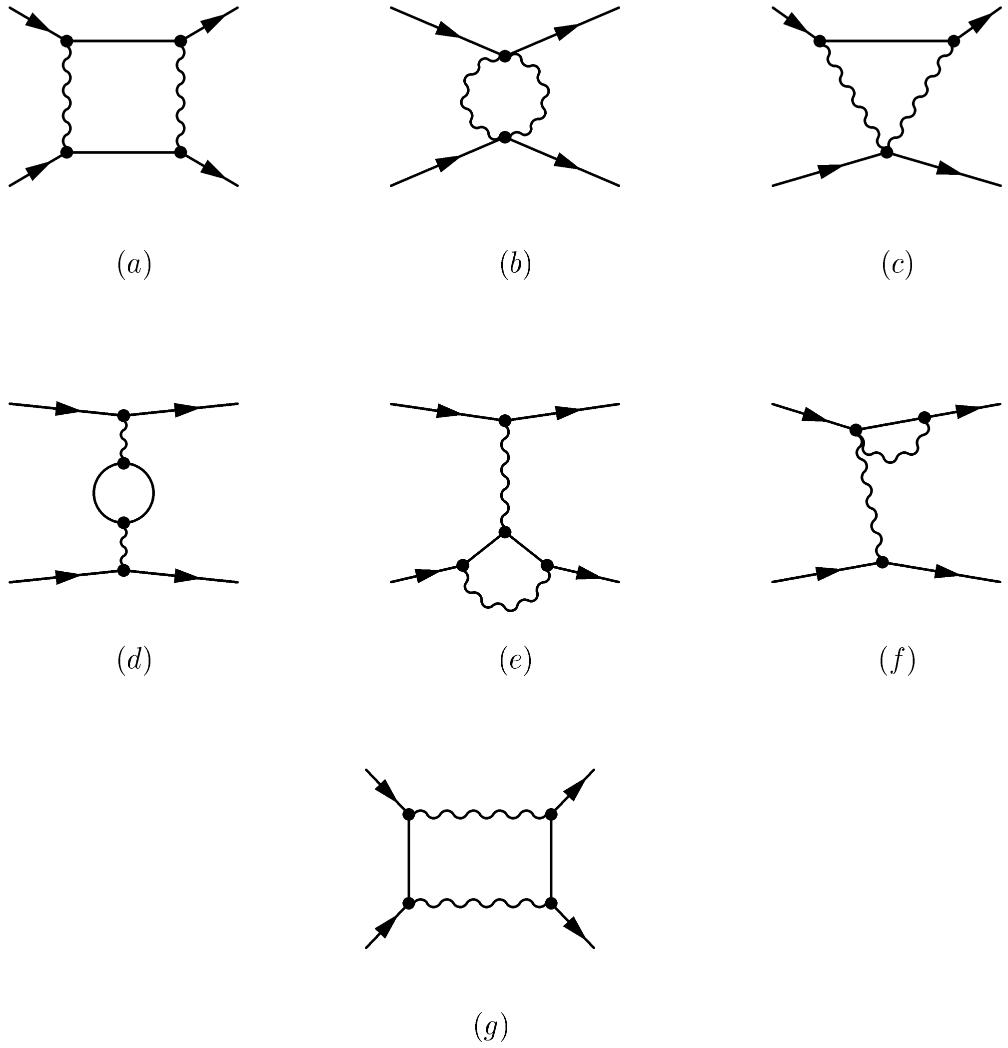


Figure 3: All possible Feynman classes of diagrams to order e^4.

composed by the graphs $3d$, $3e$ and $3f$. One of these diagrams, $3d$, has analytic behaviour as $q^2 \to 0$, because it displays only massive particles inside the loop. The contribution of this diagram in the limit $q^2 \to 0$ is proportional to $\ln(m^2)$ and therefore does not contribute to the long-distance force. The diagrams $3e$ and $3f$ have infrared divergences, but these divergences do not depend on the momentum transfer and thus do not contribute to the long-distance force. This is the usual "soft photons" situation, and it can be, for instance, treated by adding constant IR divergent counterterms [18]. For the sake of compactness, we do not consider these trivial IR divergences in what follows, since they cancel after summing up all graphs. The third group consists of the diagrams depicted in figure $3g$ and their permuted partners. They do not lead to any contribution in the limit of low momentum transfer, as they are just a quantum correction to the the tree-level graph of figure 4, and they are therefore subject to the same kinematical restrictions.

The diagram of figure 4 describes annihilation of two massive particles into a virtual photon with its subsequent decay, creating another massive pair. Obviously, the momentum transfer satisfies the energy condition $\vec{q}^2 \geq 2m^2$. Clearly, this diagram vanishes for low-momentum transfers and therefore does not contribute to the long-distance force.

A major concern, which arises when throwing away such a family of different diagrams, is the gauge-invariance of the entire set of diagrams. It is not *a priory* established whether, or not, this symmetry is preserved if we adopt this usual definition of potential.

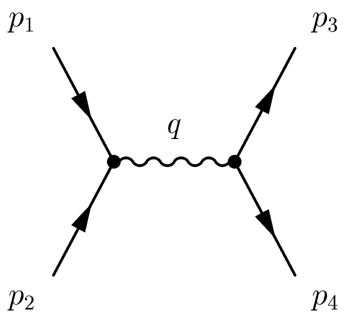


Figure 4: The t-channel process is irrelevant at the infrared limit.

4. Calculation of the relevant diagrams

Now, using the Feynman rules given above, together with the appropriate mass-shell conditions, we can derive the contributions to the scattering amplitudes from each of the relevant graphs. The mass-shell conditions for the external momenta and momentum transfer have the form:

$$\begin{aligned}
&p_1 - p_3 = q \quad \Rightarrow \quad p_1 \cdot p_3 = p_2 \cdot p_4 = m^2 - \frac{q^2}{2}\,, \\
&p_1 \cdot q = -p_2 \cdot q = -p_3 \cdot q = p_4 \cdot q = \frac{q^2}{2}\,, \\
&p_1 \cdot p_2 = p_3 \cdot p_4 = m^2 - \frac{q^2}{2}\,, \\
&p_1 \cdot p_4 = p_3 \cdot p_2 = m^2\,.
\end{aligned} \tag{4.1}$$

Now, based on our analysis of the previous section, let us consider only the diagrams $3a$–$3e$ which have essential non-analytic parts. The expression for the diagram $3b$ is the easiest one to compute, because it includes only two massless propagators.

$$\begin{aligned}
\text{fig. } 3b = 16e^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-q)^2} - 8e^4(1+\alpha) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k-q)^2} + \\
+ 4e^4(1+\alpha) \int \frac{d^4k}{(2\pi)^4} \frac{k_\lambda (k-q)^\lambda \, k_\tau (k-q)^\tau}{k^2(k-q)^2}\,.
\end{aligned} \tag{4.2}$$

The expression for the diagram $3d$ contains an additional massive propagator:

$$\begin{aligned}
\text{fig. } 3c = -2e^4 \int \frac{d^4k}{(2\pi)^4} \frac{(2p_1-k)_\lambda (p_1+p_3-k)^\lambda}{k^2(k-q)^2[(k-p_1)^2-m^2]} + \\
+ 2e^4(1+\alpha) \int \frac{d^4k}{(2\pi)^4} \frac{(p_1+p_3-k)_\lambda (k-q)^\lambda (2p_1-k)_\sigma (k-q)_\sigma}{k^2\,(k-q)^4\,[(k-p_1)^2-m^2]} +
\end{aligned}$$

$$+\,2e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{k_\lambda(p_1+p_3-k)^\lambda k_\sigma(2p_1-k)_\sigma}{k^4(k-q)^2[(k-p_1)^2-m^2]}-$$

$$-\,2e^4(1+\alpha)^2\int\frac{d^4k}{(2\pi)^4}\frac{k_\lambda(k-q)^\lambda\,(k-q)_\tau(p_1+p_3-k)^\tau k_\rho(2p_1-k)^\rho}{k^4\,(k-q)^4\,[(k-p_1)^2-m^2]}\,. \tag{4.3}$$

In all the integrals proportional to the gauge-fixing factors, $(1+\alpha)$ and $(1+\alpha)^2$, it can be checked that one term in the numerator of the integrand always cancels the massive propagators, after the mass-shell conditions are taken into account. In order to see these cancellations, one has to notice that the term $(p_1+p_3-k)_\lambda(k-q)^\lambda$ is equal to $k^2-2k\cdot p_1$, which is exactly the denominator of the massive propagator, due to the mass-shell relation $p_1^2=m^2$. In the next integrals for this graph, one can also use the relation $k\cdot(2p_1-k)=(k-p_1)^2-m^2$. One should notice that also $(p_2+p_4+k)_\sigma(k-q)^\sigma$ equals $k^2-2k\cdot p_2$. So, by using the expressions (4.1), we arrive at the following expressions for the diagrams:

$$\text{fig. } 3c=-2e^4\int\frac{d^4k}{(2\pi)^4}\frac{(2p_1-k)_\lambda(p_1+p_3-k)^\lambda}{k^2(k-q)^2[(k-p_1)^2-m^2]}-$$

$$-\,2e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{(2p_1-k)_\sigma(k-q)_\sigma}{k^2\,(k-q)^4}-$$

$$-\,2e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{k_\lambda(p_1+p_3-k)^\lambda}{k^4(k-q)^2}+$$

$$+\,2e^4(1+\alpha)^2\int\frac{d^4k}{(2\pi)^4}\frac{k_\lambda(k-q)^\lambda\,(k-q)_\tau(p_1+p_3-k)^\tau}{k^4\,(k-q)^4}\,. \tag{4.4}$$

Next graph, $(2c')$, is obtained from $(2c)$ upon the following exchange:

$$p_1\to p_2\,,\qquad p_3\to p_4\,,\qquad k\to -k\qquad\text{and}\qquad q\to -q\,,$$

which gives

$$\text{fig. } 3c\text{'}=-2e^4\int\frac{d^4k}{(2\pi)^4}\frac{(2p_2+k)_\lambda(p_2+p_+k)^\lambda}{k^2(k-q)^2[(k+p_2)^2-m^2]}+$$

$$+\,2e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{(2p_2+k)_\lambda(k-q)^\lambda}{k^2(k-q)^4}+$$

$$+\,2e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{k_\lambda(p_2+p_4-k)^\lambda}{k^4(k-q)^2}-$$

$$-\,2e^4(1+\alpha)^2\int\frac{d^4k}{(2\pi)^4}\frac{k_\lambda(k-q)^\lambda\,(k-q)_\tau(p_2+p_4+k)^\tau}{k^4(k-q)^4}\,, \tag{4.5}$$

The contribution of the "box" diagram has the form:

$$\text{fig. } 3a=e^4\int\frac{d^4k}{(2\pi)^4}\frac{(2p_1-k)_\lambda(2p_2+k)^\lambda(p_2+p_4+k)_\tau(p_1+p_3-k)^\tau}{k^2(k-q)^2[(k-p_1)^2-m^2][(k+p_2)^2-m^2]}+$$

$$
\begin{aligned}
&+e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{(2p_1-k)_\lambda(2p_2+k)^\lambda}{k^2(k-q)^4}+\\
&+e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k)^\lambda(p_1+p_3-k)^\lambda}{k^4(k-q)^2}-\\
&-2e^4(1+\alpha)^2\int\frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k)_\lambda(k-q)^\lambda(k-q)_\tau(p_1+p_3-k)^\tau}{k^4(k-q)^4}\,.
\end{aligned}
\tag{4.6}
$$

The same exchange of external momenta is used above to obtain Fig. $(2a')$, which represent the graph where the internal photons cross each other internally.

$$
\begin{aligned}
\text{fig. } 3a\text{'} = e^4&\int\frac{d^4k}{(2\pi)^4}\frac{(2p_1-k)_\lambda(k-2p_4)^\lambda(k-p_2-p_4)_\tau(p_1+p_3-k)^\tau}{k^2(k-q)^2[(k-p_1)^2-m^2][(k-p_4)^2-m^2]}+\\
&+e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{(2p_1-k)_\lambda(k-2p_4)^\lambda}{k^2(k-q)^4}+\\
&+e^4(1+\alpha)\int\frac{d^4k}{(2\pi)^4}\frac{(k-p_2-p_4+k)^\lambda(p_1+p_3-k)^\lambda}{k^4(k-q)^2}-\\
&-2e^4(1+\alpha)^2\int\frac{d^4k}{(2\pi)^4}\frac{(k-p_2-p_4)_\lambda(k-q)^\lambda(k-q)_\tau(p_1+p_3-k)^\tau}{k^4(k-q)^4}\,.
\end{aligned}
\tag{4.7}
$$

One can notice that the expressions for the gauge-dependent integrals look very similar. Our procedure to show the gauge-independence consists in grouping together all the integrals proportional to $(1+\alpha)$ and $(1+\alpha)^2$, without solving them explicitly. After that, we just make use of the on-shell conditions and perform suitable momentum redefinitions in order to check that the overall expression vanishes. So, we expand, for arbitrary α,

$$
T(q)=T_0(q)+(1+\alpha)T_1(q)+(1+\alpha)^2T_2(q)\,,
\tag{4.8}
$$

and, using the previous expressions for the diagrams, we may write T_1 and T_2 as follows:

$$
\begin{aligned}
T_1(q)=&-8\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-q)^2}-2\int\frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-(k-q))_\alpha(k-q)^\alpha}{k^2(k-q)^4}-\\
&-2\int\frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-k)_\alpha k^\alpha}{k^4(k-q)^2}+2\int\frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+(k-q))_\alpha(k-q)^\alpha}{k^2(k-q)^4}+\\
&+2\int\frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k)_\alpha k^\alpha}{k^4(k-q)^2}+\\
&+\int\frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-(k-q))_\alpha(p_2+p_4+(k-q))^\alpha}{k^2(k-q)^4}+\\
&+\int\frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-k)_\alpha(p_2+p_4+k)^\alpha}{k^4(k-q)^2}+\\
&+\int\frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-(k-q))_\alpha((k-q)-p_2-p_4)^\alpha}{k^2(k-q)^4}+\\
&+\int\frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-k)_\alpha(k-p_2-p_4)^\alpha}{k^4(k-q)^2}\,.
\end{aligned}
\tag{4.9}
$$

By analyzing these expressions, it is possible to eliminate integrals of the form

$$\int \frac{(p_1+p_3)\cdot k}{k^n(k-q)^m} \qquad \text{or} \qquad \int \frac{(p_2+p_4)\cdot k}{k^n(k-q)^m}\,,$$

because the sum will depend linearly on q_μ. One can readily see that $(p_1+p_3)\cdot q = 0$ and $(p_2+p_4)\cdot q = 0$, for $q = p_1 - p_3 = p_4 - p_2$. The same argument may be applied to similar terms, like $(p_2+p_4)\cdot(k-q)$, where, after the shift of momentum $k \to k+q$, we arrive at the same situation as before. Finally, we are left with the vanishing expression:

$$T_1(q) = e^4(1+\alpha)\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-q)^2}\cdot(-4+8-4) = 0\,. \tag{4.10}$$

Now, considering the part proportional to $(1+\alpha)^2$, and collecting all integrals we have:

$$\begin{aligned} T_2(q) = {} & 4\int \frac{d^4k}{(2\pi)^4}\frac{[k_\lambda(k-q)^\lambda]^2}{k^4(k-q)^4} + 2\int \frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-k)_\alpha(k-q)^\alpha\, k_\lambda(k-q)^\lambda}{k^4(k-q)^4} - \\ & - 2\int \frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k)_\alpha(k-q)^\alpha\, k_\lambda(k-q)^\lambda}{k^4(k-q)^4} - \\ & - \int \frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k)_\alpha(k-q)^\alpha\,(p_1+p_3-k)_\lambda(k-q)^\lambda}{k^4(k-q)^4} - \\ & - \int \frac{d^4k}{(2\pi)^4}\frac{(k-p_2-p_4)_\alpha(k-q)^\alpha(p_1+p_3-k)_\alpha(k-q)^\alpha}{k^4(k-q)^4}\,. \end{aligned} \tag{4.11}$$

Using exactly the same procedure as before, the following vanishing result follows:

$$T_2(q) = e^4(1+\alpha)^2\int \frac{d^4k}{(2\pi)^4}\frac{[k\cdot(k-q)]^2}{k^4(k-q)^4}\cdot(2-4+2) = 0\,. \tag{4.12}$$

As we can see, it is possible to demonstrate the overall cancellation of the gauge-fixing dependence in the definition of the potential. It is not necessary to solve the loop integrals explicitly; all we have to do is to search for convenient simplifications using on-shell conditions for the external legs. This relatively easy way of cancelling non-physical contributions to the potential is not so obvious when treating the same problem in Quantum Gravity, which is indeed our major concern. We believe that cancellations in SQED are, anyway, a good signal for the gravity counterpart. At least, now it is possible to set the definition of the potential as the non-relativistic limit of scattering amplitude of the 1 particle-irreducible graphs, and our method for treating the non-analytic pieces is consistent with the (expected) gauge-independence. Also, the cancellation of the gauge-dependent integrals coming from various diagrams confirms that the set of these diagrams is complete and none of them has been lost.

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5. Calculation of the physical potential

Grouping the remaining part of the scattering amplitudes in order to extract the quantum corrections to the classical Coulomb potential, we have:

$$
\begin{aligned}
T_0(q) = 8e^4 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-q)^2} - 2\int \frac{d^4k}{(2\pi)^4}\frac{(p_1+p_3-k+q))\cdot(p_1+p_3-k)}{k^2(k-q)^2[(k-p_1)^2-m^2]} - \\
-2e^4\int \frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k-q)_\lambda(p_2+p_4+k)^\lambda}{k^2(k-q)^2[(k+p_2)^2-m^2]} + \\
+e^4\int \frac{d^4k}{(2\pi)^4}\frac{(p_2+p_4+k)_\lambda(p_1+p_3-k)^\lambda(p_2+p_4+k-q)_\tau(p_1+p_3-k+q)^\tau}{k^2(k-q)^2[(k-p_1)^2-m^2][(k+p_2)^2-m^2]} + \\
+e^4\int \frac{d^4k}{(2\pi)^4}\frac{(k-p_2-p_4)_\lambda(p_1+p_3-k)^\lambda(k-q-p_2-p_4))_\tau(p_1+p_3-k+q)^\tau}{k^2(k-q)^2[(k-p_1)^2-m^2][(k-p_4)^2-m^2]} .
\end{aligned}
\tag{5.1}
$$

To extract the non-analytic terms, one can isolate the finite part of the integrals with the help of the dimensional regularization scheme. The integrals coming from 2 massless particles in the loop are relatively easy to perform; the integrals with one extra massive propagator, which are related to loops with 3 internal lines, are much more difficult. It was crucial for their solution the use of computer algebra techniques, once we are left with 2 Feynman parametric integrals, which are in general very complicated to be analytically solved. It was obtained, for these terms, the leading quantum corrections: the ones that are really important in the limit of small momentum transfer. As for the integrals with four different propagators, 2 massless and 2 massive off-shell particles, this was really a difficult task: the direct integrals over the Feynman parameters were impossible to be solved and the calculation required additional efforts.

Our way out to the problem was based upon the assumption that the static potential should not, in principle, be dependent on velocities. Then, afterwards, we might choose the most convenient external momentum configuration considering its conservation in each vertex separately. This trick was useful for the calculation of the loop integrals with four propagators. The configuration adopted is the one shown in the diagram of figure 5.

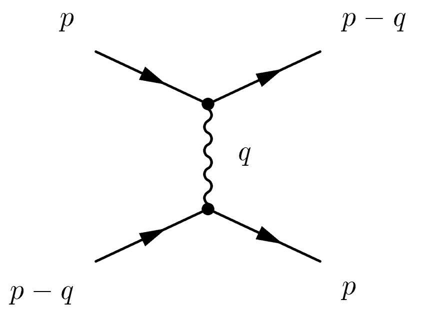

Figure 5: External momenta configuration.

For this configuration, we succeeded in finding an appropriate partial fraction decomposition for the 4-propagator integrals, and the results are given in appendix.

Another important remark is that some integrals may immediately be disregarded, because they do not present non-analytic contributions. It is rather easy to show that the integrands with k^2 or $(k-q)^2$ in the numerator do not depend on q^2, and so they have no contributions at all for the potential. This is so because these same terms appears in the denominator, representing massless propagators, and consequently simplifying them, we are left with q^2-independent integrals. Let us now put everything together, collecting the results from all integrals we have done, where we have considered only the leading IR quantum corrections. After that, we re-express the final expression in coordinate space by means of the Fourier transforms (3.1), which provides the following potential:

$$V(r) = -\frac{e^2}{r}\left(1 - \frac{3}{64\pi^2}\,\frac{e^2}{m}\,\frac{1}{r} + \frac{5}{48\pi^4}\,\frac{e^2}{m^2}\,\frac{1}{r^2}\right).$$

After restoring the powers of $\hbar$ and c, we arrive at the final expression

$$V(r) = -\frac{e^2}{r}\left(1 - \frac{3}{64\pi^2}\,e^2\,\hbar\,\frac{L_c}{r} + \frac{5}{48\pi^4}\,e^2\,\hbar\,\frac{L_c^2}{r^2}\right), \tag{5.2}$$

where $L_c = \hbar/m\,c$ is the natural length scale which shows up due to the existence of the mass parameter in SQED. We remark that such a parameter is absent in Quantum Gravity. One can see, from this potential, that, contrary to the low-energy Quantum Gravity calculations to Newton's potential, here all quantum corrections contain the factors of $\hbar$ hidden in L_c. Indeed, L_c is Compton wave length at which the quantum contributions to the potential become significant. In Quantum Gravity, there are two different scales, namely the Schwarzschild radius (for the first type corrections) and the Planck length, for quantum corrections coming from the L-type IR non-local terms.

In SQED, as we can see, there is only one scale for quantum corrections.

6. Conclusions

We have calculated the low-energy quantum corrections to the Coulomb potential in SQED. Despite its a renormalizability, SQED mimics most of the properties of an effective field theory approach for Quantum Gravity. We have checked that it is really possible to separate different scales for the theory. Then, we might say that this low-energy physics is completely independent from any high-energy renormalization parameters. As it was expected in [6], low-energy quantum predictions can be made in the framework of effective field theories. Being a renormalizable quantum field theory, SQED is its own low-energy effective theory; Nevertheless, one might hope to realize the same programme for Quantum Gravity. In particular, this concerns the gauge-independence of the quantum corrections to the potential, which has just been demonstrated for the SQED case. We remark that, unlike a previous investigation [11], our calculations concern all the relevant diagrams, even those which have massive internal lines. Indeed, for the contributions of the massless loops, the same results may be easily achieved by means of functional methods. For instance,

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since the UV and IR divergences are proportional in the massless case (momentum transfer playing the rôle of the regularization parameter in IR), one can simply apply the on-shell gauge-independence of the one-loop divergences (see, for example, [19] or simplified 1-loop proof in [15]). For the more complicated S-type non-localities, Feynman diagrams still remain the important tool, and we now hope to apply it for the Quantum Gravity case.

In this case, the theory is non-abelian and we expect that an overall infrared divergence should persist in the potential and its cancellation should be accomplished by bremsstrahlung contributions in the scattering processes.

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A. Technical details

Here, we present the loop integrals which were used throughout this work. All the integrals with two and three propagators can be performed directly by means of the Feynman parameter method (see [17] for a review on dimensional regularization).

Let us comment on the derivation of integrals with four propagators. One can always decompose the integrand into partial fractions, containing 3 propagators each, as indicated below:

$$
\begin{aligned}
\frac{q^2}{k^2(k-q)^2[(k-p)^2-m^2][(k+p-q)^2-m^2]} &= \frac{1}{k^2[(k-p)^2-m^2][(k+p-q)^2-m^2]} + \\
&+ \frac{1}{(k-q)^2[(k-p)^2-m^2][(k+p-q)^2-m^2]} - \\
&- \frac{1}{k^2(k-q)^2[(k+p-q)^2-m^2]} - \frac{1}{k^2(k-q)^2[(k-p)^2-m^2]} \,.
\end{aligned}
\tag{A.1}
$$

Integrating the first two terms, we do not obtain any non-analytical contribution, and consequently our interest is restricted to the last two fractions. They can be, in turn, calculated using Feynman parameters and MAPLE V. All the resulting integrals required for our calculation are displayed below. One has to notice that these expresions contain only the non-analytic parts of the integrals. Also, we have omitted the trivial infrared divergences, mentioned in section 4.

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-q)^2} = -\frac{i}{16\pi^2}\ln(-q^2)+\cdots \tag{A.2}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{k^2(k-q)^2} = -\frac{i}{16\pi^2}q_\mu\ln(-q^2)+\cdots \tag{A.3}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^2(k-q)^2} = \frac{i}{16\pi^2}q_\mu q_\nu\ln(-q^2)+\cdots \tag{A.4}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{k^2(k-q)^4} = \frac{i}{16\pi^2}q_\mu\frac{\ln(-q^2)}{q^2}+\cdots \tag{A.5}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^4(k-q)^4} = \frac{i}{16\pi^2}q_\mu q_\nu\frac{\ln(-q^2)}{q^4}+\cdots \tag{A.6}$$

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k-q)^2[(k-p_1)^2-m^2]} = -\frac{i}{32\pi^2m^2}\left(\ln(-q^2)+\frac{\pi^2 m}{\sqrt{-q^2}}\right)\cdots \tag{A.7}$$

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{k^2(k-q)^2[(k-p)^2-m^2]} &= \frac{i}{32\pi^2m^2}\left\{q_\mu\left(-\ln(-q^2)-\frac{1}{2}\frac{\pi^2 m}{\sqrt{-q^2}}\right)+\right.\\ &\qquad\left. +\ p_{1\mu}\ln(-q^2)\right\}\end{aligned} \tag{A.8}$$

$$\begin{aligned}&\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu}{k^2(k-q)^2[(k-p)^2-m^2][(k-p)^2-m^2]} =\\ &\qquad = \frac{-i}{32\pi^2m^2}\left\{q_\mu\left(-\frac{1}{4m^2}\frac{\pi^2 m}{\sqrt{-q^2}}-\frac{1}{2m^2}\ln(-q^2)\right)+\right.\\ &\qquad\qquad\left. +p_\mu\left(\frac{1}{2m^2}\frac{\pi^2 m}{\sqrt{-q^2}}+\frac{1}{m^2}\ln(-q^2)\right)\right\}\end{aligned} \tag{A.9}$$

$$\begin{aligned}&\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^2(k-q)^2[(k-p)^2-m^2][(k-p)^2-m^2]} =\\ &\qquad = \frac{i}{32\pi^2m^2}\left\{q_\mu q_\nu\left(-\frac{3}{8m^2}\frac{\pi^2 m}{\sqrt{-q^2}}-\frac{1}{m^2}\ln(-q^2)\right)-\right.\\ &\qquad\qquad - p_\mu p_\nu\frac{1}{m^2}\ln(-q^2)+(p_\mu q_\nu+p_\nu q_\mu)\times\\ &\qquad\qquad \times\left(\frac{1}{4m^2}\frac{\pi^2 m}{\sqrt{-q^2}}+\frac{1}{m^2}\ln(-q^2)\right)+\\ &\qquad\qquad\left. +\frac{1}{2}\eta_{\mu\nu}\ln(-q^2)\right\}\end{aligned} \tag{A.10}$$

$$\begin{aligned}&\int \frac{d^4k}{(2\pi)^4}\cdot\frac{1}{k^2(k-q)^2[(k-p)^2-m^2][(k+p-q)^2-m^2]} =\\ &\qquad = \frac{i}{32\pi^2m^2}\left\{\frac{2}{q^2}\left(\ln(q^2)+\frac{\pi^2 m}{\sqrt{-q^2}}\right)+\right. \qquad\qquad (A.11)\\ &\qquad\qquad\left. +\ \frac{1}{4m^2}\cdot\frac{\pi^2 m}{\sqrt{-q^2}}+\frac{1}{3m^2}\ln(-q^2)\right\}\end{aligned}$$

References

[1] G. 't Hooft and M. Veltman, *Ann. Inst. H. Poincaré.* **A20** (1974) 69.

[2] S. Deser and P. van Nieuwenhuizen, *One loop divergences of quantized Einstein-Maxwell fields*, *Phys. Rev.* **D 10** (1974) 401.

[3] K.S. Stelle, *Renormalization of higher derivative quantum gravity*, *Phys. Rev.* **D 16** (1977) 953.

[4] Asorey M., López J.L. and Shapiro I.L., *Some remarks on high derivative quantum gravity*, *Int. J. Mod. Phys.* **A 12** (1997) 5711.

[5] S. Weinberg, in *General relativity.* S.W. Hawking and W. Israel eds., Cambridge University Press 1979.

[6] J.F. Donoghue, *Leading quantum correction to the newtonian potential*, *Phys. Rev. Lett.* **72** (1994) 2996 [gr-qc/9310024]; *General relativity as an effective field theory: the leading quantum corrections*, *Phys. Rev.* **D 50** (1994) 3874 [gr-qc/9405057].

[7] Y. Iwasaki, *Prog. Theor. Phys.* **46** (1971) 1587.

[8] S. Weinberg, *The quantum theory of fields*, vols. I and II, Cambridge University Press 1995/1996.

[9] A. Akhundov, S. Bellucci and A. Shiekh, *Gravitational interaction to one loop in effective quantum gravity*, *Phys. Lett.* **B 395** (1997) 16 [gr-qc/9611018].

[10] H.W. Hamber and S. Liu, *Feynman rules for simplicial gravity*, *Nucl. Phys.* **B 472** (1996) 447 [hep-th/9603016].

[11] D.A. R. Dalvit and F.D. Mazzitelli, *Geodesics, gravitons and the gauge fixing problem*, *Phys. Rev.* **D 56** (1997) 7779 [hep-th/9708102].

[12] I.J. Muzinich and S. Vokos, *Long range forces in quantum gravity*, *Phys. Rev.* **D 52** (1995) 3472 [hep-th/9501083].

[13] T. Appelquist and J. Carazzone, *Infrared singularities and massive fields*, *Phys. Rev.* **D 11** (1975) 2856.

[14] R.E. Kallosh, O.V. Tarasov and I.V. Tyutin, *One loop finiteness of quantum gravity off mass shell*, *Nucl. Phys.* **B 137** (1978) 145.

[15] Barvinsky A.O., Vilkovisky G.A., *The generalized Schwinger-Dewitt technique in gauge theories and quantum gravity*, *Phys. Rep.* **119** (1985) 1.

[16] J. Sucher, *Two photon exchange force in scalar quantum electrodynamics: the asymptotic story*, *Phys. Rev.* **D 49** (1994) 4284;
G. Feinberg and J. Sucher, *The two photon exchange force between charged systems, 1. Spinless particles*, *Phys. Rev.* **D 38** (1988) 3763;

J. Sucher, *The concept of potential in quantum field theory*, hep-ph/9412388; *What is the force between electrons?*, hep-ph/9706219; *Confinement in relativistic potential models*, *Phys. Rev.* **D 51** (1995) 5965 [hep-ph/9502292].

[17] G. Leibbrandt, *Introduction to the technique of dimensional regularization*, *Rev. Mod. Phys.* **47** (1975) 849.

[18] J.D. Bjorken and S. Drell, *Relativistic quantum field theory*, McGraw-Hill, New York 1964.

[19] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective action in quantum gravity*, IOP Publishing, Bristol 1992.